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On Some Problems of P. Turán Concerning L_m Extremal Polynomials and Quadrature Formulas*

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The L_m extremal polynomials in an explicit form with respect to the weights $(1-x)^{-1/2}(1+x)^{(m-1)/2}$ and $(1-x)^{(m-1)/2}(1+x)^{-1/2}$ for even m are given. Also, an explicit representation for the Cotes numbers of the corresponding Turán quadrature formulas and their asymptotic behavior is provided.

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper let m be an even integer, w a weight (function) on $[-1, 1]$, and $\mathbf{P}_N$ the set of polynomials of degree $\leqslant N$. In what follows we denote by $c, c_1, \ldots$ the positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas.

Let

$$w^{\alpha, \beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \qquad \alpha, \beta > -1,$$

and

$$\begin{cases} v(x) := w^{(-1/2,\, -1/2)}(x), \\ u_m(x) := w^{((m-1)/2,\, (m-1)/2)}(x), \\ v_m(x) := w^{(-1/2,\, (m-1)/2)}(x), \\ w_m(x) := w^{((m-1)/2,\, -1/2)}(x). \end{cases} \tag{1.1}$$

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As we know, for each $n \in \mathbb{N}$ there exists a unique polynomial

$$P_n(w, m; x) = x^n + \cdots$$

for which

$$\int_{-1}^{1} P_n(w, m; x)^m \, w(x) \, dx = \min_{P = x^n + \cdots} \int_{-1}^{1} P(x)^m \, w(x) \, dx; \tag{1.2}$$

meanwhile, $P_n(w, m; x)$ has the zeros $x_{kn} = x_{kn}(w)$, $k = 1, 2, ..., n$, satisfying

$$1 = x_{0n} > x_{1n} > x_{2n} > \cdots > x_{nn} > x_{x+1, n} = -1. \tag{1.3}$$

As P. Turán pointed out in [10, p. 46], little is known about $P_n(w, m; x)$ for $m \geqslant 4$, apart from the well-known fact that

$$P_n(v, m; x) = 2^{1-n} T_n(x), \tag{1.4}$$

where $T_n(x)$ stands for the nth Chebyshev polynomial of the first kind. So he raised the two problems connected to this direction [10, p. 73].

Problem 74. Give the minimizing polynomials of (1.2) with $m = 4$ in an explicit form for weights other than $v(x)$.

Problem 75. Give an asymptotic representation of the minimizing polynomials of (1.2) with $m = 4$, valid on $[-1, 1]$, for a weight other than $v(x)$.

In fact, we also know (see, say, [6]) that

$$P_n(u_m, m; x) = 2^{-n} U_n(x), \tag{1.5}$$

where $U_n(x)$ stands for the nth Chebyshev polynomial of the second kind.

The first aim of the present paper is to give other solutions of these two problems.

THEOREM 1. *Let*

$$V_n(\cos\theta) = \frac{\cos[(2n+1)\,\theta/2]}{\cos(\theta/2)} \tag{1.6}$$

and

$$W_n(\cos\theta) = \frac{\sin[(2n+1)\,\theta/2]}{\sin(\theta/2)}. \tag{1.7}$$

Then

$$P_n(v_m, m; x) = 2^{-n} V_n(x) \tag{1.8}$$

and

$$P_n(w_m, m; x) = 2^{-n} W_n(x). \tag{1.9}$$

Closely related to the extremal problem (1.2) is a Gaussian quadrature formula. If we rewrite the weight w as

$$w(x) = (1-x)^p\,(1+x)^q\,u(x), \tag{1.10}$$

where p and q are nonnegative integers and $u(x)$ is a weight on $[-1, 1]$, then according to [2, Theorem 4] Eq. (1.2) admits the Gaussian quadrature formula ($x_k = x_{kn}(x)$, $k = 1, 2, ..., n$),

$$\int_{-1}^{1} f(x)\,u(x)\,dx = \sum_{k=0}^{n+1} \sum_{j=0}^{\mu_k} \lambda_{jk}(u, p, q)\,f^{(j)}(x_k), \tag{1.11}$$

which is exact for all $f \in \mathbf{P}_{mn+p+q-1}$, where

$$\mu_k = \begin{cases} m-2, & 1 \leqslant k \leqslant n, \\ p-1, & k = 0, \\ q-1, & k = n+1, \end{cases} \tag{1.12}$$

and $\lambda_{jk}(u, p, q) := \lambda_{jkm}(u, p, q) := \lambda_{jkmn}(u, p, q)$ are called Cotes numbers. For simplicity write $\lambda_{jk}(u) := \lambda_{jk}(u, 0, 0)$. For this direction Turán also posed [10, p. 47].

Problem 26. Give an explicit formula for $\lambda_{jkmn}(v)$ and determine its asymptotic behavior as $n \to \infty$.

In [5] we gave an answer to this problem. In [6, 7] we also got solutions of the same problem for the cases

$$p = q = \frac{m}{2}, \qquad w(x) = v(x)$$

and

$$p = q = 0, \qquad w(x) = u_m(x),$$

respectively. In general, for each integer r, $0 \leqslant r \leqslant m/2$, the extremal problem (1.2) with $w = v_m$ admits the Gaussian quadrature formula corresponding to the case

$$p = 0, \qquad q = \frac{m}{2} - r, \qquad w(x) = (1+x)^r \, v(x).$$

Of course, particularly interesting are the cases corresponding to $r = 0$ and $r = m/2$. The second aim of this paper is to provide answers to the same problem for the four cases:

$$p = 0, \qquad q = \frac{m}{2}, \qquad w(x) = v(x), \tag{1.13}$$

$$p = q = 0, \qquad w(x) = v_m(x), \tag{1.14}$$

$$p = \frac{m}{2}, \qquad q = 0, \qquad w(x) = v(x), \tag{1.15}$$

$$p = q = 0, \qquad w(x) = w_m(x). \tag{1.16}$$

In order to state these results we introduce the notation ($x_k = x_{kn}(v_m)$, $k = 1, 2, ..., n$):

$$\Pi_{mn}(x) := (1+x)^{m/2} \, V_n(x)^m,$$

$$n_k := \begin{cases} 1, & 1 \leqslant k \leqslant n, \\ \frac{1}{2}, & k = n+1, \end{cases}$$

$$d_{km} := \Pi_{mn}^{(n_k m)}(x_k) = \begin{cases} m!(1+x_k)^{m/2} \, V_n'(x_k)^m, & 1 \leqslant k \leqslant n, \\ (m/2)! \; V_n(-1)^m, & k = n+1, \end{cases} \tag{1.17}$$

$$L_{km}(x) := L_{kmn}(x) := \frac{(n_k m)! \; \Pi_{mn}(x)}{d_{km}(x - x_k)^{n_k m}}, \qquad k = 1, ..., n+1, \tag{1.18}$$

$$b_{ik} := b_{ikm} := b_{ikmn} := \frac{1}{i!} \, [L_{km}(x)^{-1}]^{(i)}_{x = x_k}, \;\; k = 1, ..., n+1, \;\; i = 0, 1, ... \tag{1.19}$$

Then we have

Theorem 2. *Let (1.3) be the zeros of $V_n(x)$ and let*

$$m_k := n_k(m-2).$$

Then the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\, v(x)\, dx = \sum_{k=1}^{n+1} \sum_{j=0}^{m_k} \lambda_{jkm}(v, 0, m/2)\, f^{(j)}(x_k) \tag{1.20}$$

holds for all $f \in \mathbf{P}_{mn+m/2-1}$, *where for each* j, $0 \leqslant j \leqslant m_k$, *and for each* k, $1 \leqslant k \leqslant n+1$,

$$\begin{cases} \lambda_{m_k, k, m}(v, 0, m/2) = \dfrac{2^{m/2}(m-2)!\, \pi n_k}{[(m-2)!!]^2\, (2n+1)\, d_{k, m-2}}, \\ \lambda_{m_k+1, k, m}(v, 0, m/2) = 0, \end{cases} \tag{1.21}$$

$$\lambda_{jkm}(v, 0, m/2) = \lambda_{j, k, m-2}(v, 0, (m-2)/2) + \frac{(m_k-1)!\, \lambda_{m_k, k, m}(v, 0, m/2)\, b_{m_k-j, k, m-2}}{(j-1)!}, \quad m \geqslant 4. \tag{1.22}$$

Moreover,

$$\lambda_{jkm}(v, 0, m/2) \sim \begin{cases} \dfrac{(1-x_k^2)^{j/2}}{n^{j+1}}, & 1 \leqslant k \leqslant n, \quad j \in M := \{0, 2, 4, ..., m-2\}, \\ \dfrac{1}{n^{2j+1}}, & k = n+1, \quad j = 0, 1, ..., \dfrac{m-2}{2}, \end{cases} \tag{1.23}$$

and

$$|\lambda_{jkm}(v, 0, m/2)| \leqslant c\, \frac{(1-x_k^2)^{(j-1)/2}}{n^{j+2}}, \qquad 1 \leqslant k \leqslant n, \qquad j \notin M. \tag{1.24}$$

THEOREM 3. *Let* (1.3) *be the zeros of* $V_n(x)$. *Then the Gaussian quadrature formula*

$$\int_{-1}^{1} f(x)\, v_m(x)\, dx = \sum_{k=1}^{n} \sum_{j=0}^{m-2} \lambda_{jkm}(v_m)\, f^{(j)}(x_k) \tag{1.25}$$

holds for all $f \in \mathbf{P}_{mn-1}$, *where for each* j, $0 \leqslant j \leqslant m-2$, *and for each* k, $1 \leqslant k \leqslant n$,

$$\lambda_{jkm}(v_m) = \sum_{i=j}^{m-2} \binom{i}{j} \frac{(m/2)!}{(m/2+j-i)!}\, \lambda_{ikm}(v, 0, m/2)(1+x_k)^{m/2+j-i}. \tag{1.26}$$

Moreover,

$$\begin{cases} \lambda_{jkm}(v_m) \sim \dfrac{(1-x_k^2)^{j/2}\, (1+x_k)^{m/2}}{n^{j+1}}, & j \in M, \\ |\lambda_{jkm}(v_m)| \leqslant c\, \dfrac{(1-x_k^2)^{(j-1)/2}\, (1+x_k)^{m/2}}{n^{j+2}}, & j \notin M. \end{cases} \tag{1.27}$$

THEOREM 4. *Let* (1.3) *be the zeros of* $W_n(x)$. *Then*

$$\int_{-1}^{1} f(x)\, v(x)\, dx = \sum_{k=0}^{n} \sum_{j=0}^{m_k} \lambda_{jkm}(v, m/2, 0)\, f^{(j)}(x_k), \quad f \in \mathbf{P}_{mn+m/2-1}, \tag{1.28}$$

and

$$\int_{-1}^{1} f(x)\, w_m(x)\, dx = \sum_{k=1}^{n} \sum_{j=0}^{m-2} \lambda_{jkm}(w_m)\, f^{(j)}(x_k), \qquad f \in \mathbf{P}_{mn-1}, \tag{1.29}$$

where

$$\lambda_{jkm}(v, m/2, 0) = (-1)^j\, \lambda_{j,\, n+1-k,\, m}(v, 0, m/2), \qquad 0 \leqslant j \leqslant m_k, \quad 0 \leqslant k \leqslant n, \tag{1.30}$$

$$\lambda_{jkm}(w_m) = (-1)^j\, \lambda_{j,\, n+1-k,\, m}(v_m), \qquad 0 \leqslant j \leqslant m-2, \quad 1 \leqslant k \leqslant n. \tag{1.31}$$

We introduce the notation $f[x_1^j, ..., x_n^j, x]$ for the divided difference at the points $x_1 \geqslant \cdots \geqslant x_n$ and x, where x_k^j means that the point x_k is repeated j times. Bojanov [1] established the following quadrature formula, which is an extension of a quadrature formula given by Micchelli and Rivlin [4]:

THEOREM A. *Assume that* $x_k = x_{kn}(w)$, $k = 1, 2, ..., n$, *and for each* j, $0 \leqslant j \leqslant m-2$, *the quadrature formula*

$$\int_{-1}^{1} f(x)\, P_n(w, m; x)^j\, w(x)\, dx = \sum_{k=1}^{n} c_{jkm}(w)\, f(x_k) \tag{1.32}$$

holds for all $f \in \mathbf{P}_{n-1}$. *Then the quadrature formula*

$$\int_{-1}^{1} f(x)\, w(x)\, dx = \sum_{k=1}^{n} \sum_{j=0}^{m-2} c_{jkm}(w)\, f[x_1^j, ..., x_n^j, x_k] \tag{1.33}$$

holds for all $f \in \mathbf{P}_{mn-1}$.

The third aim of the present paper is to establish such quadrature formulas for the weights $w = v_m$ and $w = w_m$.

THEOREM 5. *For* $n \geqslant m/2 - 1$ *we have the quadrature formula* (1.33) *with the weight* $w = v_m$ *and* $w = w_m$, *respectively, where for each* j, $0 \leqslant j \leqslant m-2$, *and each* k, $1 \leqslant k \leqslant n$,

$$c_{jkm}(v_m) = \begin{cases} \dfrac{2^{j/2-jn+1} j!\, \pi (1+x_k)^{(m-j)/2}}{(j!!)^2 (2n+1)}, & j \in M, \\ \dfrac{2^{m/2-jn-1} (j+1)!\, \pi}{[(j+1)!!]^2\, V_n'(x_k)} \displaystyle\sum_{i=0}^{(m-j-3)/2} \dfrac{(m-j-3-2i)!\, (1+x_k)^i}{2^i [(m-j-3-2i)!!]^2}, & j \notin M \end{cases} \tag{1.34}$$

and

$$c_{jkm}(w_m) = (-1)^{jn}\, c_{j,\, n+1-k,\, m}(v_m). \tag{1.35}$$

We give some auxiliary lemmas in the next section and the proofs of the theorems in the last section.

2. AUXILIARY LEMMAS

To prove our theorems we need several lemmas.

LEMMA 1. *For $f \in \mathbf{P}_{n-1}$ we have*

$$\int_{-1}^{1} f(x)\, T_n(x)^j\, v(x)\, dx = \begin{cases} \dfrac{j!}{(j!!)^2} \displaystyle\int_{-1}^{1} f(x)\, v(x)\, dx = \dfrac{j!\, \pi}{(j!!)^2\, n} \sum_{k=1}^{n} f(x_k), & j \in M, \\ 0, & j \notin M, \end{cases} \tag{2.1}$$

where $x_k = \cos[(2k-1)\, \pi/(2n)]$, $k = 1, 2, ..., n$.

Proof. For $j \in M$, Lemma 2 in [1] says

$$\int_{-1}^{1} f(x)\, T_n(x)^j\, v(x)\, dx = \frac{j!\, \pi}{(j!!)^2\, n} \sum_{k=1}^{n} f(x_k),$$

which, together with its special case when $j = 0$, gives (2.1). For $j \notin M$ the relation (2.1) can be found, say in [1, p. 355]. ∎

Now we use an idea of the proof of Theorem 4.1 in [9, p. 58] to derive an extension of that theorem.

LEMMA 2. *We have*

$$P_{2n}(w^{(\alpha,\alpha)}, m; x) = 2^{-n} P_n(w^{(\alpha,-1/2)}, m; 2x^2-1) \tag{2.2}$$
$$= (-2)^{-n} P_n(w^{(-1/2,\alpha)}, m; 1-2x^2),$$
$$P_{2n+1}(w^{(\alpha,\alpha)}, m; x) = 2^{-n} xP_n(w^{(\alpha,(m-1)/2)}, m; 2x^2-1) \tag{2.3}$$
$$= (-2)^{-n} xP_n(w^{((m-1)/2,\alpha)}, m; 1-2x^2).$$

Proof. We give the proof of (2.2) only; the proof of (2.3) is similar. The second equality of (2.2) follows from

$$P_n(w(-\cdot), m; -x) = (-1)^n P_n(w, m; x), \tag{2.4}$$

which may be directly derived by (1.2). In order to prove the first equality of (2.2) by means of the characterization theorem of L_m approximation it is enough to show

$$\int_{-1}^{1} P_n(w^{(\alpha,-1/2)}, m; 2x^2-1)^{m-1} Q(x)(1-x^2)^\alpha\, dx = 0, \qquad Q \in \mathbf{P}_{2n-1}. \tag{2.5}$$

Since it is trivial for the odd polynomials Q, it is sufficient to show (2.5) for the even polynomials Q. In this case we can write $Q(x) = r(2x^2-1)$, $r \in \mathbf{P}_{n-1}$. Then by making the change of variable $t = 2x^2-1$, we get

$$\int_{-1}^{1} P_n(w^{(\alpha,-1/2)}, m; 2x^2-1)^{m-1} Q(x)(1-x^2)^\alpha\, dx$$
$$= 2^{-1/2-\alpha} \int_{-1}^{1} P_n(w^{(\alpha,-1/2)}, m; t)^{m-1} r(t)\, w^{(\alpha,-1/2)}(t)\, dt = 0. \quad \blacksquare$$

As usual, we use the notation

$$\ell_{kn}(w; x) = \frac{P_n(w, m; x)}{P_n'(w, m; x_{kn}(w))(x - x_{kn}(w))}, \qquad k = 1, 2, ..., n.$$

LEMMA 3. *We have*

$$V_n(2x^2-1) = \frac{T_{2n+1}(x)}{x}, \tag{2.6}$$

$$W_n(2x^2-1) = U_{2n}(x), \tag{2.7}$$

$$W_n(-x) = (-1)^n V_n(x), \tag{2.8}$$

$$x_{kn}(w_m) = -x_{n+1-k,n}(v_m), \qquad k = 1, 2, ..., n, \tag{2.9}$$

$$\ell_{kn}(w_m; -x) = \ell_{n+1-k,n}(v_m; x), \qquad k = 1, 2, ..., n. \tag{2.10}$$

Proof. Equations (2.6) and (2.7) may be obtained by setting $\cos\theta = 2x^2-1$ in (1.6) and (1.7), respectively. Equation (2.8) directly follows from (2.4). Finally, (2.9) and (2.10) may be derived from (2.8). ▮

LEMMA 4. *Let $x_k = x_{kn}(v_m)$, $k=1, 2, ..., n$, and*

$$a_{ik} := \sum_{\substack{\nu=1\\ \nu\neq k}}^{n+1} \frac{n_\nu}{(x_\nu - x_k)^i}, \qquad i=1, 2, ..., \qquad 1\leqslant k\leqslant n+1.$$

Then

$$b_{ikm} = \frac{m}{i}\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu, k, m}, \qquad i=1, 2, ..., \qquad 1\leqslant k\leqslant n+1.$$

Proof. Let k, $1\leqslant k\leqslant n+1$, be fixed. It is easy to check that

$$\frac{L'_{km}(x)}{L_{km}(x)} = -\sum_{\substack{\nu=1\\ \nu\neq k}}^{n+1} \frac{n_\nu m}{x_\nu - x},$$

which implies

$$a_{ik} = \frac{1}{m(i-1)!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}\right]^{(i-1)}_{x=x_k}.$$

Hence applying the Newton–Leibniz rule, it follows from (1.19) that

$$\begin{aligned}\frac{m}{i}&\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu, k, m}\\ &= \frac{m}{i}\sum_{\nu=1}^{i} \frac{1}{m(\nu-1)!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}\right]^{(\nu-1)}_{x=x_k} \frac{1}{(i-\nu)!}\,[L_{km}(x)^{-1}]^{(i-\nu)}_{x=x_k}\\ &= \frac{1}{i!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}\,L_{km}(x)^{-1}\right]^{(i-1)}_{x=x_k} = \frac{1}{i!}\,[L_{km}(x)^{-1}]^{(i)}_{x=x_k} = b_{ikm}.\end{aligned}$$ ▮

3. PROOFS OF THEOREMS

3.1. *Proof of Theorem* 1. Setting $\alpha = (m-1)/2$ in (2.2) and using (1.5) and (2.7), we obtain

$$P_n(w_m, m; 2x^2-1) = 2^n\, P_{2n}(u_m, m; x) = 2^{-n}\, U_{2n}(x) = 2^{-n}\, W_n(2x^2-1),$$

which means (1.9). Then by (2.4), (1.9), and (2.8),

$$P_n(v_m, m; x) = (-1)^n P_n(w_m, m; -x) = (-2)^{-n} W_n(-x) = 2^{-n} V_n(x). \quad \blacksquare$$

Remark. Equations (1.8) and (1.9) may also be obtained by setting $\alpha = -1/2$ in (2.3).

3.2. *Proof of Theorem* 5. We note that by (1.8) and (1.32),

$$c_{jkm}(v_m) = \int_{-1}^{1} [2^{-n} V_n(x)]^j \ell_k(v_m; x)\, v_m(x)\, dx, \quad 0 \leqslant j \leqslant m-2, \quad 1 \leqslant k \leqslant n. \tag{3.1}$$

Then by making the change of variable $x = 2t^2 - 1$ and applying (2.6),

$$\begin{aligned} c_{jkm}(v_m) &= 2^{(m+2)/2 - jn} \int_0^1 V_n(2t^2-1)^j \ell_k(v_m; 2t^2-1)\, t^m v(t)\, dt \\ &= 2^{m/2 - jn} \int_{-1}^1 V_n(2t^2-1)^j \ell_k(v_m; 2t^2-1)\, t^m v(t)\, dt \\ &= 2^{m/2 - jn} \int_{-1}^1 \ell_k(v_m; 2t^2-1)\, t^{m-j} T_{2n+1}(t)^j v(t)\, dt. \end{aligned}$$

We distinguish two cases.

Case 1: $j \in M$. In this case, by Lemma 1

$$\begin{aligned} c_{jkm}(v_m) &= 2^{m/2-jn} \Bigg[\int_{-1}^1 \ell_k(v_m; 2t^2-1)\, t^2 (t^{m-j-2} - t_k^{m-j-2})\, T_{2n+1}(t)^j v(t)\, dt \\ &\quad + t_k^{m-j-2} \int_{-1}^1 \ell_k(v_m; 2t^2-1)\, t^2\, T_{2n+1}(t)^j v(t)\, dt \Bigg] \\ &= 2^{m/2-jn} \left[\int_{-1}^1 \frac{t(t^{m-j-2} - t_k^{m-j-2})}{2V_n'(x_k)(t^2 - t_k^2)} T_{2n+1}(t)^{j+1} v(t)\, dt + \frac{2(j!)\,\pi t_k^{m-j}}{(j!!)^2 (2n+1)} \right] \\ &= \frac{2^{j/2 - jn+1} j!\, \pi (1+x_k)^{(m-j)/2}}{(j!!)^2 (2n+1)}. \end{aligned}$$

Case 2: $j \notin M$. In this case, by Lemma 1 and the formula [3, 3.621-2, p. 369]

$$\int_{-1}^1 x^{2i} v(x)\, dx = \int_0^\pi \cos^{2i}\theta\, d\theta = \frac{(2i)!\,\pi}{[(2i)!!]^2}, \qquad i = 0, 1, ...,$$

we get

$$
\begin{aligned}
c_{jkm}(v_m) &= 2^{m/2-jn} \int_{-1}^{1} \ell_k(v_m; 2t^2-1)\, t(t^{m-j-1} - t_k^{m-j-1})\, T_{2n+1}(t)^j\, v(t)\, dt \\
&= 2^{m/2-jn} \int_{-1}^{1} \frac{t^{m-j-1} - t_k^{m-j-1}}{2V_n'(x_k)(t^2-t_k^2)}\, T_{2n+1}(t)^{j+1}\, v(t)\, dt \\
&= 2^{m/2-jn} \frac{(j+1)!}{[(j+1)!!]^2} \int_{-1}^{1} \frac{t^{m-j-1} - t_k^{m-j-1}}{2V_n'(x_k)(t^2-t_k^2)}\, v(t)\, dt \\
&= \frac{2^{m/2-jn-1}(j+1)!}{[(j+1)!!]^2\, V_n'(x_k)} \sum_{i=0}^{(m-j-3)/2} t_k^{2i} \int_{-1}^{1} t^{m-j-3-2i} v(t)\, dt \\
&= \frac{2^{m/2-jn-1}(j+1)!\, \pi}{[(j+1)!!]^2\, V_n'(x_k)} \sum_{i=0}^{(m-j-3)/2} \frac{(m-j-3-2i)!\, (1+x_k)^i}{2^i[(m-j-3-2i)!!]^2}.
\end{aligned}
$$

This proves (1.34).

To prove (1.35) we make the change of variable $x = -t$, and use (1.9), (1.32), (2.8), and (2.10) to get

$$
\begin{aligned}
c_{jkm}(w_m) &= \int_{-1}^{1} [2^{-n}\, W_n(x)]^j\, \ell_k(w_m; x)\, w_m(x)\, dx \\
&= \int_{-1}^{1} [2^{-n}\, W_n(-t)]^j\, \ell_k(w_m; -t)\, w_m(-t)\, dt \\
&= (-1)^{jn} \int_{-1}^{1} [2^{-n}\, V_n(t)]^j\, \ell_{n+1-k}(v_m; t)\, v_m(t)\, dt \\
&= (-1)^{jn}\, c_{j,\, n+1-k,\, m}(v_m). \quad \blacksquare
\end{aligned}
$$

3.3. *Proof of Theorem* 2. Here we use the idea of [6]. Let k, $1 \leqslant k \leqslant n+1$, be fixed.

First let us prove (1.21), since (1.20) is obvious. To this end we make $\Omega_{n+1}(x) = (1+x)\, V_n(x)$. It is easy to check that

$$
f(x) = \frac{\Pi_{m-2,\, n}(x)\, \Omega_{n+1}(x)}{d_{k,\, m-2}\Omega'_{n+1}(x_k)(x-x_k)}
$$

satisfies the interpolatory conditions

$$
f^{(\mu)}(x_\nu) = \delta_{\mu,\, m_k}\, \delta_{\nu,\, k}, \qquad \mu = 0, 1, ..., m_k, \qquad \nu = 1, ..., n+1.
$$

In fact, it is sufficient to show that

$$f^{(m_k)}(x_k)=1.$$

By the Newton–Leibniz rule and (1.17) this is indeed the case:

$$f^{(m_k)}(x_k)=\left[\frac{\Pi_{m-2,n}(x)\,\Omega_{n+1}(x)}{d_{k,m-2}\Omega'_{n+1}(x_k)(x-x_k)}\right]^{(m_k)}_{x=x_k}=\frac{1}{d_{k,m-2}}\,\Pi^{(m_k)}_{m-2,n}(x_k)=1.$$

Substituting $f\in \mathbf{P}_{(m-1)n+m/2-1}$ into (1.20) gives

$$\lambda_{m_k,k,m}=\int_{-1}^{1}\frac{\Pi_{m-2,n}(x)\,\Omega_{n+1}(x)}{d_{k,m-2}\Omega'_{n+1}(x_k)(x-x_k)}\,v(x)\,dx,\qquad k=1,2,...,n+1. \tag{3.2}$$

We distinguish two cases.

Case 1: $1\leqslant k\leqslant n$. By means of (3.2), (3.1), and (1.34), we have

$$\begin{aligned}\lambda_{m_k,k,m}&=\frac{1}{d_{k,m-2}(1+x_k)}\int_{-1}^{1}V_n(x)^{m-2}\,\ell_k(v_m;x)\,v_m(x)\,dx\\&=\frac{2^{(m-2)n}c_{m-2,k,m}(v_m)}{d_{k,m-2}(1+x_k)}=\frac{2^{m/2}(m-2)!\,\pi}{[(m-2)!!]^2\,(2n+1)\,d_{k,m-2}}.\end{aligned}$$

Case 2: $k=n+1$. By making the change of variable $x=2t^2-1$, it follows from (3.2), (2.6), and (2.1) that

$$\begin{aligned}\lambda_{(m-2)/2,n+1,m}&=\frac{1}{d_{n+1,m-2}V_n(-1)}\int_{-1}^{1}V_n(x)^{m-1}\,v_{m-2}(x)\,dx\\&=\frac{2^{(m-2)/2}}{d_{n+1,m-2}V_n(-1)}\int_{-1}^{1}V_n(2t^2-1)^{m-1}\,t^{m-2}v(t)\,dt\\&=\frac{2^{(m-2)/2}}{d_{n+1,m-2}V_n(-1)}\int_{-1}^{1}V_n(2t^2-1)\,T_{2n+1}(x)^{m-2}\,v(t)\,dt\\&=\frac{2^{(m-2)/2}}{d_{n+1,m-2}V_n(-1)}\cdot\frac{(m-2)!\,\pi V_n(-1)}{[(m-2)!!]^2\,(2n+1)}\\&=\frac{2^{m/2}(m-2)!\,\pi n_{n+1}}{[(m-2)!!]^2\,(2n+1)\,d_{n+1,m-2}}.\end{aligned}$$

Next let us prove (1.22). By [8, (1.4)] we conclude that

$$f(x)=\frac{1}{j!}\sum_{i=0}^{m_k-1-j}b_{i,k,m-2}(x-x_k)^{i+j}\,L_{k,m-2}(x) \tag{3.3}$$

satisfies the interpolatory conditions

$$f^{(\mu)}(x_\nu)=\delta_{j\mu}\,\delta_{k\nu}, \qquad \mu=0, 1, ..., m_k-1, \qquad \nu=1, ..., n+1.$$

Substituting $f\in \mathbf{P}_{(m-2)\,n+m/2-2}$ into (20) and in (20) replacing m by $m-2$, we have

$$\lambda_{j,\,k,\,m-2}=\int_{-1}^{1} f(x)\,v(x)\,dx=\lambda_{jkm}+\sum_{\nu=1}^{n+1}\lambda_{m_\nu,\,\nu,\,m}f^{(m_\nu)}(x_\nu), \tag{3.4}$$

where $\lambda_{jkr}=\lambda_{jkr}(v, 0, r/2)$.

We distinguish two cases.

Case 1: $\nu=k$. By (1.17) and (1.18) we have $L_{k,\,m-2}(x_k)=1$. Using (3.3) and (1.19) and applying the Newton–Leibniz rule twice, we obtain

$$\begin{aligned}
f^{(m_k)}(x_k)&=\frac{1}{j!}\sum_{i=0}^{m_k-1-j} b_{i,\,k,\,m-2}[(x-x_k)^{i+j}\,L_{k,\,m-2}(x)]^{(m_k)}_{x=x_k}\\
&=\frac{(m_k)!}{j!}\sum_{i=0}^{m_k-1-j} b_{i,\,k,\,m-2}\,\frac{L^{(m_k-j-i)}_{k,\,m-2}(x_k)}{(m_k-j-i)!}\\
&=\binom{m_k}{j}\sum_{i=0}^{m_k-1-j}\binom{m_k-j}{i}\,[L_{k,\,m-2}(x)^{-1}]^{(i)}_{x=x_k}\,L^{(m_k-j-i)}_{k,\,m-2}(x_k)\\
&=\binom{m_k}{j}\{[1]^{(m_k-j)}_{x=x_k}-(m_k-j)!\,b_{m_k-j,\,k,\,m-2}\}\\
&=-\frac{(m_k)!\,b_{m_k-j,\,k,\,m-2}}{j!}.
\end{aligned}\tag{3.5}$$

Case 2: $\nu\neq k$. Using (3.3) and applying the Newton–Leibniz rule again, we have

$$\begin{aligned}
&\sum_{\nu\neq k}\lambda_{m_\nu,\,\nu,\,m}f^{(m_\nu)}(x_\nu)\\
&\quad=\frac{1}{j!}\sum_{\nu\neq k}\lambda_{m_\nu,\,\nu,\,m}\sum_{i=0}^{m_k-1-j} b_{i,\,k,\,m-2}[(x-x_k)^{i+j}\,L_{k,\,m-2}(x)]^{(m_\nu)}_{x=x_\nu}\\
&\quad=\frac{1}{j!}\sum_{i=0}^{m_k-1-j} b_{i,\,k,\,m-2}\sum_{\nu\neq k}\lambda_{m_\nu,\,\nu,\,m}[(x-x_k)^{i+j}\,L_{k,\,m-2}(x)]^{(m_\nu)}_{x=x_\nu}\\
&\quad=\frac{1}{j!}\sum_{i=0}^{m_k-1-j} b_{i,\,k,\,m-2}\sum_{\nu\neq k}\lambda_{m_\nu,\,\nu,\,m}(x_\nu-x_k)^{i+j}\,L^{(m_\nu)}_{k,\,m-2}(x_\nu).
\end{aligned}\tag{3.6}$$

According to (1.17) and (1.18) we see

$$L_{k,m-2}^{(m_v)}(x_v)=\frac{d_{v,m-2}m_k!}{d_{k,m-2}(x_v-x_k)^{m_k}},\qquad v\neq k. \tag{3.7}$$

Substituting (1.21) and (3.7) into (3.6) and applying Lemma 4, we obtain

$$\begin{aligned}\sum_{v\neq k}\lambda_{m_v,v,m}f^{(m_v)}(x_v)
&=\frac{2^{m/2}(m-2)!\,\pi m_k!}{j![(m-2)!!]^2\,(2n+1)\,d_{k,m-2}}\sum_{i=0}^{m_k-1-j}b_{i,k,m-2}\sum_{v\neq k}\frac{n_v}{(x_v-x_k)^{m_k-i-j}}\\
&=\frac{m_k!\lambda_{m_k,k,m}}{j!\,n_k}\sum_{i=0}^{m_k-1-j}b_{i,k,m-2}a_{m_k-j-i,k}\\
&=\frac{(m_k-1)!\,(m_k-j)\,\lambda_{m_k,k,m}b_{m_k-j,k,m-2}}{j!},\end{aligned}$$

which, coupled with (3.4) and (3.5), gives (1.22).

Finally, let us prove (1.23) and (1.24). By [5, (21)],

$$|[\ell_k(x)^{-m}]_{x=x_k}^{(i)}|\leqslant c_1\frac{n^{2[i/2]}}{(1-x_k^2)^{[(i+1)\,2]}},\qquad 1\leqslant k\leqslant n. \tag{3.8}$$

Meanwhile it is easy to see that

$$\left|\left[\left(\frac{1+x}{1+x_k}\right)^{-m/2}\right]_{x=x_k}^{(i)}\right|\leqslant c_2(1+x_k)^{-i},\qquad 1\leqslant k\leqslant n. \tag{3.9}$$

Then by the Newton–Leibniz rule,

$$\begin{aligned}|b_{ikm}|&=\left|\frac{1}{i!}\left[\left(\frac{1+x}{1+x_k}\right)^{-m/2}\ell_k(x)^{-m}\right]_{x=x_k}^{(i)}\right|\\
&\leqslant c\frac{n^{2[i/2]}}{(1-x_k^2)^{[(i+1)/2]}},\qquad 1\leqslant k\leqslant n.\end{aligned} \tag{3.10}$$

Here we use the relation

$$(1+x_k)^{-1}\leqslant c_3n^2.$$

By (1.6) we obtain

$$|V_n'(x_k)|\leqslant c\frac{n}{(1-x_k)^{1/2}\,(1+x_k)} \tag{3.11}$$

and

$$V_n(-1) = (-1)^n (2n+1). \tag{3.12}$$

By (1.8) we have $V_n(x) = 2^n P_n(v_2, 2; x)$ and hence according to the proof of [9, Theorem 7.32.1, pp. 163–164] we conclude that $|V_n(x)| \leqslant |V_n(-1)|$. Thus

$$|b_{i,n+1,m}| = \left\| \left[\left(\frac{V_n(x)}{V_n(-1)} \right)^{-m} \right]^{(i)}_{x=-1} \right| \leqslant cn^{2i}. \tag{3.13}$$

Then by (1.17), (1.21), (3.10), and (3.11) we get, for each k, $1 \leqslant k \leqslant n$,

$$\begin{aligned} |\lambda_{jkm}| &\leqslant |\lambda_{j,k,m-2}| + c\lambda_{m_k,k,m} |b_{m_k-j,k,m-2}| \\ &\leqslant |\lambda_{j,k,m-2}| + \frac{c(1-x_k^2)^{(m-2)/2}\, n^{2[(m-j-2)/2]}}{n^{m-1}(1-x_k^2)^{[(m-j-1)/2]}} \\ &\leqslant |\lambda_{j,k,m-2}| + \frac{c(1-x_k^2)^{[j/2]}}{n^{2[(j+1)/2]+1}}. \end{aligned}$$

By induction it follows from (1.21) that

$$|\lambda_{jkm}| \leqslant \frac{c(1-x_k^2)^{[j/2]}}{n^{2[(j+1)/2]+1}}.$$

Using (1.17), (1.21), (3.12), and (3.13) we have

$$|\lambda_{j,n+1,m}| \leqslant |\lambda_{j,n+1,m-2}| + c\lambda_{m_{n+1},n+1,m} |b_{m_{n+1}-j,n+1,m-2}| \leqslant \frac{c}{n^{2j+1}}.$$

To estimate the lower bounds of λ_{jkm} we note that by [8, (2.8) and (2.9)]

$$b_{jk} > 0, \qquad j \in M; \qquad b_{j,n+1} > 0, \qquad j = 0, 1, ..., \frac{m-2}{2}.$$

Thus by (1.17), (1.21), (1.22), (3.11), and (3.12) we even have

$$\begin{aligned} \lambda_{jkm} \geqslant \lambda_{j,k,m-2} \geqslant \lambda_{j,k,j+2} &= \frac{2^{(j+2)/2} j!\, \pi}{(j!!)^2 (2n+1)\, d_{k,j}} \\ &\geqslant \frac{c(1-x_k^2)^{j/2}}{n^{j+1}}, \qquad 1 \leqslant k \leqslant n, \qquad j \in M, \end{aligned}$$

and

$$\lambda_{j,\,n+1,\,m} \geqslant \lambda_{j,\,n+1,\,2j+2} \geqslant \frac{2^j (2j)!\,\pi}{[(2j)!!]^2\,(2n+1)\,d_{n+1,\,2j}}$$

$$\geqslant \frac{c}{n^{2j+1}}, \qquad j = 0, 1, ..., \frac{m-2}{2}. \quad \blacksquare$$

3.4. *Proof of Theorem* 3. Let $A_{jk} \in \mathbf{P}_{mn-1}$, $0 \leqslant j \leqslant m-1$, $1 \leqslant k \leqslant n$, satisfy the interpolatory conditions

$$A_{jk}^{(\mu)}(x_\nu) = \delta_{j\mu}\,\delta_{k\nu}, \qquad \mu = 0, 1, ..., m-1, \qquad \nu = 1, 2, ..., n.$$

Inserting $f = A_{jk}$ into (1.25) and using (1.20) yield

$$\begin{aligned}
\lambda_{jkm}(v_m) &= \int_{-1}^{1} A_{jk}(x)\, v_m(x)\, dx \\
&= \int_{-1}^{1} A_{jk}(x)(1+x)^{m/2}\, v(x)\, dx \\
&= \sum_{i=0}^{m-2} \lambda_{ik}(v, 0, m/2)[A_{jk}(x)(1+x)^{m/2}]^{(i)}_{x=x_k} \\
&= \sum_{i=j}^{m-2} \binom{i}{j} \frac{(m/2)!}{(m/2+j-i)!} \lambda_{ik}(v, 0, m/2)(1+x_k)^{m/2+j-i}.
\end{aligned}$$

This proves (1.26).

With the help of (1.26) the estimations of the upper bounds of $\lambda_{jkm}(v_m)$ follow directly from (1.23) and (1.24).

To estimate the lower bounds of $\lambda_{jkm}(v_m)$ we use an inequality given by the author [8, (2.1)]:

$$\lambda_{jkm}(w) \geqslant c h_k^{j-i} \lambda_{ikm}(w) > 0, \qquad i > j, \qquad i, j \in M,$$

where $h_1 = |x_1 - x_2|$, $h_n = |x_n - x_{n-1}|$, and $h_k = \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}$, $2 \leqslant k \leqslant n-1$. Clearly, $h_k \sim (1 - x_k^2)^{1/2}/n$. Hence by (1.26) and (1.23) for $j \in M$

$$\begin{aligned}
\lambda_{jkm}(v_m) &\geqslant c h_k^{j-(m-2)} \lambda_{m-2,\,k,\,m}(v_m) \\
&\geqslant c h_k^{j-(m-2)} \lambda_{m-2,\,k,\,m}(v, 0, m/2)(1+x_k)^{m/2} \\
&\geqslant c\,\frac{(1-x_k^2)^{j/2}\,(1+x_k)^{m/2}}{n^{j+1}}. \quad \blacksquare
\end{aligned}$$

3.5. *Proof of Theorem* 4. Equations (1.28) and (1.29) are obvious. Now let us prove (1.31) only, since the proof of (1.30) may be derived in a similar way.

Let $A_{jk}(w_m)$, $A_{j,n+1-k}(v_m) \in \mathbf{P}_{mn-1}$, $0 \leqslant j \leqslant m-1$, $1 \leqslant k \leqslant n$, satisfy the interpolatory conditions:

$$A_{jk}^{(\mu)}(w_m; x_\nu(w_m)) = A_{j,n+1-k}^{(\mu)}(v_m; x_{n+1-\nu}(v_m)) = \delta_{j\mu}\,\delta_{k\nu},$$
$$\mu = 0, 1, ..., m-1, \qquad \nu = 1, 2, ..., n.$$

Then by (2.9),

$$A_{j,n+1-k}^{(\mu)}(v_m; -x_\nu(w_m))$$
$$= (-1)^\mu A_{j,n+1-k}^{(\mu)}(v_m; x_{n+1-\nu}(v_m)) = (-1)^j \delta_{j\mu}\,\delta_{k\nu},$$
$$\mu = 0, 1, ..., m-1, \qquad \nu = 1, 2, ..., n,$$

which means

$$A_{jk}(w_m; x) = (-1)^j A_{j,n+1-k}(v_m; -x). \tag{3.14}$$

By making the change of variable $x = -t$, it follows from (1.29), (3.14), and (1.25) that

$$\lambda_{jkm}(w_m) = \int_{-1}^{1} A_{jk}(w_m; x)\, w_m(x)\, dx$$
$$= (-1)^j \int_{-1}^{1} A_{j,n+1-k}(v_m; -x)\, w_m(x)\, dx$$
$$= (-1)^j \int_{-1}^{1} A_{j,n+1-k}(v_m; t)\, v_m(t)\, dt = (-1)^j \lambda_{j,n+1-k,m}(v_m). \quad \blacksquare$$

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